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THE KERNEL AND BARGAINING SET FOR CONVEX GAMES

M. Maschler, B. Peleg and L. S. Shapley

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PREFACE

This Memorandum is one of a series of technical papers dealing with topics in the mathematical theory of n-person games. Convex games are competitive situations in which there are strong incentives toward formation of large coalitions; exampl can be found in both commercial and diplomatic contexts.

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ABSTRACT

A convex game is characterized by increasing marginal utility for coalition membership as coalitions grow larger. The core of any n-person game is the set of outcomes that cannot be profitably blocked by any coalition. For the case of convex games, two other solution concepts—the kernel and the bargaining set—prove to be closely related to the core. In fact, it is shown that the kernel lies in the relative interior of the core and that the bargaining set coincides with the core. Similar results were obtained previously (see [9]) for yet two other solution concepts, the value and the von Neumann-Morgenstern stable sets.

CONTENTS

THE KERNEL AND BARGAINING SET FOR CONVEX GAMES

1. INTRODUCTION

The basic papers dealing with the kernel $\mathcal{K}$ of a cooperative game are [2], [5] and [6]. The basic papers dealing with the bargaining set $\mathcal{M}_1^{(i)}$ are [3], [4] and [7]. For intuitive justification of the bargaining set as a solution concept, the reader is referred to [1].

Although the kernel of a game and the core of a game are two different concepts, it has been proved in [5] that if a game has a nonempty core, its kernel for the grand coalition must intersect the core; however, it may contain points also outside the core. The bargaining set for the grand coalition always contains the core and, again, it may contain points outside the core.

A task which naturally arises is to sharpen these results for games whose core is in some sense "nice."

Convex games were introduced in [9], where it was shown that these are precisely the games for which the core is regular.

Interpreting "nice" as regular, we shall accomplish the task by showing that, for the grand coalition, the kernel of a convex game lies in the relative interior of the core--i.e., its interior when regarded as a point set in the linear manifold of least dimension which contains it. We shall also show that the bargaining set $\mathcal{M}_1^{(i)}$ of a convex game coincides

with the core, and hence, as shown in [9] with the unique von Neumann-Morgenstern solution.

Although all the definitions as well as the results taken from other papers will be fully stated, it is advisable that the reader make himself familiar at least with the relevant parts of [2], [4], [6]. [8] and [9].

2. THE KERNEL OF A CONVEX GAME

An n-person cooperative game $(N; v)$, where $N = \{1, 2, \ldots, n\}$ is its set of players and v is its characteristic function, is called <u>convex</u> if v satisfies

$$(2.1) \qquad v(\phi) = 0,$$

$$(2.2) \qquad v(A) + v(B) \leq v(A \cup B) + v(A \cap B) \quad \text{all} \quad A, B \subset N.$$

Convex games were introduced and studied in [9], where it has been shown that they have nonempty cores. Moreover, it has been shown that these are precisely the games whose core is <u>regular</u>.[1]

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[1] The core of an n-person game $(N; v)$ is the set of all n-tuples $x = (x_1, \ldots, x_n)$ such that $x(N) = v(N)$ and $x(S) \geq v(S)$ for all coalitions S. Here $x(S)$ is a short notation for $\Sigma_{i \in S}\, x_i$. The notation $\dot{x}(S)$ will be used throughout this paper. $x(\phi)$ is defined to be equal to zero.

A core $\mathbf{C}$ of an n-person game $(N; v)$ is called regular if it is not empty and if, in addition,

$$\mathbf{C}_S \cap \mathbf{C}_T \subset \mathbf{C}_{S \cup T} \cap \mathbf{C}_{S \cap T}.$$

Here $\mathbf{C}_A = \mathbf{C} \cap \{x = (x_1, \ldots, x_n) | x(A) = v(A)\}$, $A = S, T$. In particular $\mathbf{C}_S \neq \phi$ for all S if $\mathbf{C}$ is regular. Thus, a regular core is quite "large" since it touches all the (n-2)-faces

Convex games are superadditive but not necessarily monotonic.(2) However, if the characteristic function of a game satisfies

$$(2.3) \qquad v(\{i\}) = 0, \quad i = 1, 2, \ldots, n,$$

then monotonicity follows from superadditivity.

Note that the core, the bargaining set and the kernel of a game are relative invariants with respect to strategic equivalence,(3) and that convexity is invariant under strategic equivalence. For this reason we shall assume (2.3) in some of the proofs and this assumption will entail no loss of generality.

Let $x = (x_1, x_2, \ldots, x_n)$ be an n-tuple of real numbers. We define the <u>excess</u> of a coalition S with respect to x to be

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of the simplex of imputations. In [9] it is shown that a regular core always contains the (Shapley) value, and is the unique von Neumann-Morgenstern solution of the game.

(2)A game (N; v) is called <u>monotonic</u> if v satisfies $v(S) \leq v(T)$ whenever $S \subset T$.

(3)I.e., they undergo the transformation $x \to ax + \alpha$ when $v(S)$ is replaced by $av(S) + \alpha(S)$ for each coalition S. Here a is a real positive constant, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of real constants, and $\alpha(S)$ is a short notation for $\Sigma_{i \in S}\, \alpha_i$.

$$e(S, x) = v(S) - x(S). \tag{2.4}$$

LEMMA 2.1. *An n-person game (N; v) is convex if and only if for any arbitrarily chosen fixed n-tuple* x,

$$e(S, x) + e(T, x) \leq e(S \cup T, x) + e(S \cap T, x) \tag{2.5}$$

for each pair of coalitions S, T.

The proof is immediate.

Let $\mathcal{C}$ be a given collection of coalitions in a convex n-person game (N; v), and let x be an arbitrarily chosen n-tuple of real numbers. Let $\mathcal{D}(\mathcal{C}, x)$ be the collection consisting of those coalitions in $\mathcal{C}$ whose excess with respect to x is maximal; i.e.,

$$\mathcal{D}(\mathcal{C}, x) = \{S \mid S \in \mathcal{C} \text{ and } e(S, x) \geq e(T, x) \text{ all } T \in \mathcal{C}\}. \tag{2.6}$$

LEMMA 2.2. *Under these conditions,* $\mathcal{D}(\mathcal{C}, x)$ *is "nearly" closed under unions and intersections; namely,*

$$S, T \in \mathcal{D}(\mathcal{C}, x) \Rightarrow S \cup T \in \mathcal{D}(\mathcal{C}, x), \tag{2.7}$$

$$S, T \in \mathcal{D}(\mathcal{C}, x) \Rightarrow S \cap T \in \mathcal{D}(\mathcal{C}, x), \tag{2.8}$$

provided that both $S \cup T$ *and* $S \cap T$ *belong to* $\mathcal{C}$.

The proof is an immediate consequence of Lemma 2.1 and the definitions.

We shall write $\mathcal{D}(x)$ instead of $\mathcal{D}(\mathcal{C}, x)$ if $\mathcal{C}$ is the set of all the coalitions of the game except N and ϕ. We shall write $\mathcal{E}(x)$ instead of $\mathcal{D}(\mathcal{C}, x)$ if $\mathcal{C}$ is the set of all the coalitions of the game.

COROLLARY 2.3. $\mathcal{E}(x)$ is closed under unions and intersections.

An n-tuple $x = (x_1, x_2, \ldots, x_n)$ is called an imputation in an n-person game $\Gamma = (N; v)$ if $x(\{i\}) \geq v(\{i\})$ for all i, $i = 1, 2, \ldots, n$ and $x(N) = v(N)$. An imputation x belongs to the core of the game Γ, if and only if $e(S, x) \leq 0$ for every coalition S. The core of a game will be denoted by $\mathbf{C}$ or by $\mathbf{C}(\Gamma)$. An imputation x is said to belong to the kernel of the game Γ for the grand coalition N if for every ordered pair of players (k, ℓ),

$$(2.9) \qquad \max_{S: k \in S, \ell \notin S} e(S, x) \leq \max_{S: \ell \in S, k \notin S} e(S, x) \quad \text{or} \quad x_\ell = v(\{\ell\}).$$

The kernel for the grand coalition will be denoted in this paper by $\mathcal{K}$ or $\mathcal{K}(\Gamma)$.(4) It has been shown in general that $\mathcal{K} \neq \phi$ and that $\mathcal{K} \cap \mathbf{C} \neq \phi$ if $\mathbf{C} \neq \phi$ (see [2], [5]). In [9] it is shown that for convex games $\mathbf{C} \neq \phi$. Both $\mathbf{C}$ and $\mathcal{K}$ are closed sets.

THEOREM 2.4. The kernel for the grand coalition of a convex game is contained in the core.

(4) If coalition-structures other than $\{N\}$ are also being considered, the usual notation is slightly different.

Proof. Let $\Gamma = (N; v)$ be a convex game. Without loss of generality we may assume that (2.3) holds; in which case Γ is a monotonic game. For monotonic games satisfying (2.3) it has been proved in [6] that if $x \in \mathcal{K}$ then $\mathcal{D}(x)$ (see notation prior to Corollary 2.3) has the following property:[5]

> If a coalition in $\mathcal{D}(x)$ contains a player k and does not contain a player ℓ then another coalition exists in $\mathcal{D}(x)$ which contains player ℓ and does not contain player k.

It follows from this property that[6]

$$(2.10) \qquad \cap_{S:S\in\mathcal{D}(x)} S = \phi$$

$$(2.11) \qquad \cup_{S:S\in\mathcal{D}(x)} S = N$$

for $n \geq 2$, whenever $x \in \mathcal{K}$, because $\mathcal{D}(x)$ is not empty and its members are proper nonempty subsets of N.

The theorem certainly holds for 1-person games. Assuming $n \geq 2$ and applying Lemma 2.2 repeatedly to unions and intersections of members of $\mathcal{D}(x)$, one concludes that either there exist two coalitions S_1 and T_1 in $\mathcal{D}(x)$ such that $S_1 \cap T_1 = \phi$, or there exist two coalitions S_2 and T_2 in

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(5) In the terminology of [6], no player is "separated out" by $\mathcal{D}(x)$. $\mathcal{D}(x)$ is denoted in [6] as $\mathcal{D}(N, x)$.

(6) (2.10) holds for a much larger class of games (see [6]).

$\mathcal{D}(x)$ such that $S_2 \cup T_2 = N$. In view of the fact that $e(N, x) = e(\phi, x) = 0$, it follows from Lemma 2.1 and from the meaning of $\mathcal{D}(x)$ that $e(S, x) \leq 0$ for every coalition of the game. Consequently x belongs to the core of the game. This concludes the proof.

Let us now ask under what conditions the kernel of a convex game can intersect the boundary of the core, when the core is being regarded as a point-set in the $n - 1$ dimensional hyperplane $x(N) = v(N)$. Assume, therefore, that $x \in \mathcal{K}$ and $e(S, x) = 0$ for a coalition S other than N and ϕ. By Theorem 2.4 it follows that the coalitions in $\mathcal{D}(x)$ have excess equal to 0 and therefore, in this case,

$$(2.12) \qquad \mathcal{E}(x) = \mathcal{D}(x) \cup \{N\} \cup \{\phi\}.$$

DEFINITION 2.5. The partition $\{T_1, T_2, \dots, T_u\}$ of N induced by a given collection of coalitions $\mathcal{D}$ is the set of equivalence classes $T_1, T_2, \dots, T_u$ such that two players k and ℓ belong to the same equivalence class if and only if they appear simultaneously in the coalitions of $\mathcal{D}$; i.e., if and only if

$$(2.13) \qquad k \in S \in \mathcal{D} \iff \ell \in S \in \mathcal{D}.$$

Clearly, one obtains the same partition if $\mathcal{D}$ is replaced by $\mathcal{D} \cup \{N\} \cup \{\phi\}$.

We shall now show that the equivalence classes $T_1, T_2, \ldots, T_u$ of the partition induced by $\mathcal{D}(x)$ (or by $\mathcal{E}(x)$, see (2.12)) are themselves elements of $\mathcal{D}(x)$, provided that $n \geq 2$.

Since $\mathcal{D}(x)$ and therefore $\{T_1, T_2, \ldots, T_u\}$ are invariant under strategic equivalence,[7] we may assume that (N; v) satisfies (2.3) and is, therefore, a monotonic game.

Consider the intersection $I(\ell)$ of all the coalitions in $\mathcal{E}(x)$ which contain a player ℓ in an equivalence class T_j. Clearly $I(\ell) \supset T_j$. By Corollary 2.3, $I(\ell) \in \mathcal{E}(x)$. By (2.11) and because $n \geq 2$, $I(\ell) \neq N$ and $I(\ell) \neq \phi$; therefore

$$I(\ell) \in \mathcal{D}(x). \tag{2.14}$$

If a player k existed in $I(\ell) - T_j$ then there would exist a coalition in $\mathcal{D}(x)$ containing k and not ℓ, whereas every coalition in $\mathcal{D}(x)$ containing player ℓ would also contain player k. This is impossible by the property of $\mathcal{D}(x)$ mentioned prior to (2.10). We conclude that $I(\ell) = T_j$, and, by (2.14), $T_j \in \mathcal{D}(x)$. Thus, $e(T_1, x) = e(T_2, x) = \ldots = e(T_u, x) = 0$, and since $\{T_1, T_2, \ldots, T_u\}$ is a partition of N it follows from (2.4) that

$$v(T_1) + v(T_2) + \ldots + v(T_u) = v(N). \tag{2.15}$$

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[7]When x undergoes the transformation mentioned in footnote (3).

DEFINITION 2.6. A game $(N; v)$ is said to be decomposable into the games $(T_1; v/T_1)$, $(T_2; v/T_2)$, ..., $(T_u; v/T_u)$, where $\{T_1, T_2, \ldots, T_u\}$, $u \geq 2$, is a partition of N and where v/T_j, $j = 1, 2, \ldots, u$, is the restriction of the characteristic function v to the subsets of T_j, if for each coalition S, $S \subset N$,

$$(2.16) \qquad v(S) = v(S \cap T_1) + v(S \cap T_2) + \ldots + v(S \cap T_u).$$

The notion of decomposition was introduced by J. von Neumann and O. Morgenstern in [10]. We shall refer to the games $(T_j; v/T_j)$ as components of $(N; v)$. It is easily seen from (2.2) that a decomposable game is convex if and only if all its components are convex. Thus, many nontrivial decomposable games exist for $n \geq 4$; none of them, however, is "strictly convex" in the sense of [9].[8]

LEMMA 2.7. *A convex game* $(N; v)$ *is decomposable into* $(T_1; v/T_1)$, $(T_2; v/T_2)$, ..., $(T_u; v/T_u)$, *where* $\{T_1, T_2, \ldots, T_u\}$ *is a partition of* N, *if and only if*

$$(2.17) \qquad v(N) = v(T_1) + v(T_2) + \ldots + v(T_u).$$

Proof.[9] Clearly, (2.16) implies (2.17). Suppose

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[8] A game is called strictly convex if the strict inequality holds in (2.2) whenever the four sets A, B, $A \cup B$, $A \cap B$ are all distinct.

[9] Several proofs can be given. The proof presented here is, perhaps, the most elegant one. It was pointed out to one of the authors by Y. Kannai.

(2.17) holds. By convexity,

$$v(S) + v(T_1) \leq v(S \cap T_1) + v(S \cup T_1)$$
$$v(S \cup T_1) + v(T_2) \leq v(S \cap T_2) + v(S \cup T_1 \cup T_2)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$v(S \cup T_1 \cup \cdots \cup T_{u-1}) + v(T_u) \leq v(S \cap T_u) + v(S \cup T_1 \cup \cdots \cup T_u).$$

Adding these inequalities to the equation (2.17), we obtain $v(S) \leq v(T_1 \cap S) + v(T_2 \cap S) + \cdots + v(T_u \cap S)$. But equality must hold because $(N; v)$ is a superadditive game. This concludes the proof.

A consequence of Theorem 2.4, Lemma 2.7 and the preceding discussion leading to (2.15) is the following.

THEOREM 2.8. The kernel for the grand coalition of an indecomposable n-person convex game lies in the interior of the core.

This has been proved for $n \geq 2$. It certainly holds if $n = 1$.

The converse statement is also true in view of the next lemma.

LEMMA 2.9. The core of a decomposable game is the cartesian product of the cores of its components. Therefore it is not of full dimension, and each of its points is a boundary point.

Proof. Let $x \in \mathbf{C}(\Gamma)$, where $\Gamma = (N; v)$ is decomposable into $\Gamma_1 = (T_1; v/T_1)$, $\Gamma_2 = (T_2; v/T_2)$, $\ldots$, $\Gamma_u = (T_u; v/T_u)$,

where $\{T_1, T_2, \ldots, T_u\}$ is a partition of N, $u \geq 2$. By (2.17), $0 = v(N) - x(N) = [v(T_1) - x(T_1)] + [v(T_2) - x(T_2)] + \ldots + [v(T_u) - x(T_u)] = e(T_1, x) + e(T_2, x) + \ldots + e(T_u, x)$. Therefore, $e(T_1, x) = e(T_2, x) = \ldots = e(T_u, x) = 0$ and x/T_j – the restriction of x into the coordinates which belong to T_j – belong to $\mathcal{C}(\Gamma_j)$, $j = 1, \ldots, u$. And conversely, if $x/T_j \in \mathcal{C}(\Gamma_j)$, $j = 1, 2, \ldots, u$, then, by (2.16) and (2.4), for each subset S of N, $e(S, x) = e(S \cap T_1, x) + e(S \cap T_2, x) + \ldots + e(S \cap T_u, x) \leq 0$ and therefore $x \in \mathcal{C}(\Gamma)$.

It has been proved in [8] that, in general, the kernel for the grand coalition of a decomposable game is not equal to the cartesian product of the kernels for the component games, because a transfer (see [8], [10]) may take place. For convex games, however, no transfer is possible.

LEMMA 2.10. The kernel for the grand coalition of a decomposable convex game is the cartesian product of the kernels for the grand coalitions of the component games.

Proof. We use the notation of the previous proof. It has been shown in [8] that $\mathcal{K}(\Gamma) \supset \mathcal{K}(\Gamma_1) \times \mathcal{K}(\Gamma_2) \times \ldots \times \mathcal{K}(\Gamma_u)$. If $x \in \mathcal{K}(\Gamma)$ then $x \in \mathcal{C}(\Gamma)$ (Theorem 2.4) and therefore x/T_j is an imputation in Γ_j, $j = 1, 2, \ldots, u$. Thus there is no transfer and, as shown in [8], $x/T_j \in \mathcal{K}(\Gamma_j)$. This concludes the proof.

We can now sharpen the result stated in Theorems 2.4 and 2.8.

THEOREM 2.11. The kernel for the grand coalition of a convex game lies in the relative interior of the core.

<u>Proof</u>. Suppose Γ is decomposable into $\Gamma_1, \Gamma_2, \ldots, \Gamma_u$ and none of the Γ_j's is further decomposable. If $x \in \mathcal{K}(\Gamma)$ then $x/T_j \in \mathcal{K}(\Gamma_j)$ and, by Theorem 2.8, x/T_j lies in the interior of $\mathbf{C}(\Gamma_j)$, where $\mathbf{C}(\Gamma_j)$ is being regarded as a point-set in the linear manifold spanned by the imputations of Γ_j, $j = 1, 2, \ldots, u$. The rest of the proof now follows easily.

3. THE BARGAINING SET $\mathcal{M}_1^{(i)}$ FOR A CONVEX GAME

Let x be an imputation in a game (N; v). An objection of a player k against a player ℓ, with respect to x, is a pair ($\hat{y}$; C); where C is a coalition containing player k and not containing player ℓ, $\hat{y}$ is a vector whose indices are the members of C, $\hat{y}(C) = v(C)$, and $\hat{y}_i > x_i$ for each i in C. A counter objection to the above objection is a pair ($\hat{z}$; D), where D is a coalition containing player ℓ and not containing player k and $\hat{z}$ is a vector whose indices are members of D, $\hat{z}(D) = v(D)$, $z_i \geq y_i$ for $i \in D \cap C$, and $z_i \geq x_i$ for $i \in D - C$.

An imputation x is said to belong to the bargaining set $\mathcal{M}_1^{(i)}$ for the grand coalition if for any objection of one player against another there exists a counter objection to this objection.[10] Clearly, $\mathcal{M}_1^{(i)}$ contains the core, because if $x \in \mathcal{C}$ no objections are possible. In this section we shall show that for convex games $\mathcal{M}_1^{(i)} = \mathcal{C}$. Since $\mathcal{M}_1^{(i)} \supset \mathcal{K}$ (see [2]), this result will furnish another proof of Theorem 2.4.

THEOREM 3.1. *The bargaining set $\mathcal{M}_1^{(i)}$ for the grand coalition of a convex game coincides with the core of the game.*

Proof. All we have to show is that if x is an imputation not in the core then $x \notin \mathcal{M}_1^{(i)}$.

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[10] The notation is slightly different if coalition-structures other than {N} are also being considered.

Let $s(x)$ be the excess of the coalitions in $\mathcal{E}(x)$ (see notation prior to Corollary 2.3). Since $x \notin \mathcal{C}$,

$$s(x) > 0. \tag{3.1}$$

Consequently

$$N \notin \mathcal{E}(x) \quad \phi \notin \mathcal{E}(x). \tag{3.2}$$

By Corollary 2.3, we have $Q_1 \in \mathcal{E}(x)$, $Q_2 \in \mathcal{E}(x)$, where

$$Q_1 = \bigcup_{S: S \in \mathcal{E}(x)} S \neq N, \tag{3.3}$$

$$Q_2 = \bigcap_{S: S \in \mathcal{E}(x)} S \neq \phi. \tag{3.4}$$

Thus, there exists a player k in Q_2 and a player ℓ in $N - Q_1$. We shall conclude the proof by showing that k has an objection against ℓ with respect to x which cannot be countered.

Let C be any coalition in $\mathcal{E}(x)$. Then $k \in C$ and $\ell \notin C$. It has been shown in [4] that an objection $(\mathfrak{y}; C)$ which cannot be countered exists with respect to x if (and only if) the following conditions hold:

(i) $e(R, x) < 0$ whenever $\ell \in R$, $R \cap C = \phi$,

(ii) $e(C, x) > e(R, x)$ whenever $\ell \in R$ and $R \cap C = C - \{k\}$,

(iii) The game $(C - \{k\}; v_C^*)$ has a full dimensional core.

Here,

$$(3.5)\qquad v_C^*(S) = \begin{cases} e(C, x) & \text{if } S = C - \{k\}, \\ \max(0, \max\limits_{\substack{R:R\cap C=S \\ \ell\in R}} e(R,x)) & \text{if } S \subset C - \{k\}, \\ & \quad S \neq C - \{k\},\ \phi, \\ 0 & \text{if } S = \phi. \end{cases}$$

We shall show that these conditions hold. Condition (ii) holds because $R \notin \mathcal{E}(x)$ due to the fact that $\ell \in R$. If $R \cap C = \phi$ and $\ell \in R$, then, by Lemma 2.1,

$$(3.6)\qquad s(x) + e(R, x) = e(C, x) + e(R, x) \leq e(C \cup R) < s(x).$$

This implies $e(R, x) < 0$ as required by condition (i).

Finally, in order to prove (iii), we shall construct a convex game $(C - \{k\};\ v_C^{**})$ whose characteristic function v_C^{**} satisfies

$$(3.7)\qquad v_C^{**}(C - \{k\}) < v_C^*(C - \{k\}).$$

$$(3.8)\qquad v_C^{**}(S) \geq v_C^*(S) \text{ whenever } S \subset C - \{k\},\ S \neq C - \{k\}.$$

Being convex, this new game would have a nonempty core, and because of (3.7) and (3.8), $(C - \{k\};\ v_C^*)$ would have a full dimensional core.

To this end, let us define

$$(3.9) \qquad v_C^{**}(S) = \max(0, \max_{\substack{R: R\cap C \subset S \\ \text{and } \ell \in R}} e(R, x))$$

for all S, $S \subset C - \{k\}$. Relation (3.7) now follows from (3.1), (3.5), the fact that $C \in \mathcal{E}(x)$, and the fact that $R \notin \mathcal{E}(x)$ if $\ell \in R$. Relation (3.8) is established by comparing (3.5) with (3.9).

Note that $v_C^{**}(S) \geq 0$ and that the game $(C - \{k\}; v_C^{**})$ is monotonic (see footnote (2)). Therefore, for $A, B \subset C - \{k\}$, the relation

$$(3.10) \qquad v_C^{**}(A) + v_C^{**}(B) \leq v_C^{**}(A \cup B) + v_C^{**}(A \cap B)$$

holds whenever either $v_C^{**}(A) = 0$ or $v_C^{**}(B) = 0$. If $v_C^{**}(A) > 0$ and $v_C^{**}(B) > 0$ then, by (3.9), coalitions R_A and R_B exist in the original game such that $\ell \in R_A$, $\ell \in R_B$, $R_A \cap C \subset A$, $R_B \cap C \subset B$, $v_C^{**}(A) = e(R_A, x)$ and $v_C^{**}(B) = e(R_B, x)$. It now follows from Lemma 2.1 and (3.9) that $v_C^{**}(A) + v_C^{**}(B) = e(R_A, x) + e(R_B, x) \leq e(R_A \cup R_B, x) + e(R_A \cap R_B, x) \leq v_C^{**}(A \cup B) + v_C^{**}(A \cap B)$. We have therefore showed that (3.10) holds also in this case. Thus, $(C - \{k\}; v_C^{**})$ is, indeed, a convex game. This concludes the proof.

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10. ABSTRACT

In game theory, a convex game is a competitive situation characterized by increasing marginal utility for coalition membership as coalitions grow larger. The core of any n-person game is the set of outcomes that cannot profitably be blocked by any coalition. For the case of convex games, two other solution concepts--the kernel and the bargaining set--prove to be closely related to the core. The kernel lies in the relative interior of the core, and the bargaining set coincides with the core. RM-4571-PR, which introduced the convex game, showed that the core is similarly related to two other solution concepts: the value solution is the center of gravity of the extreme points of the core, and the Von Neumann-Morgenstern stable set solution coincides with the core.

11. KEY WORDS

Game theory
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